

# UNMIXED $d$ -UNIFORM $r$ -PARTITE HYPERGRAPHS

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**ABSTRACT.** In this paper, we characterize all unmixed  $d$ -uniform  $r$ -partite hypergraphs under a certain condition. Also we give a necessary condition for unmixedness in  $d$ -uniform hypergraphs with a perfect matching of size  $n$ . Finally we give a sufficient condition for unmixedness in  $d$ -uniform hypergraphs with a perfect matching.

## 1. Introduction

Unmixedness is one of the most important concepts in theory of graphs and hypergraphs with nice and interesting algebraic and geometric interpretations (for instance see [4], [6], [7], [9], [10], [13]). According to this, characterization of special classes of unmixed graphs has been noteworthy in recent years. G. Ravindra in [8] and R. H. Villarreal in [11] have characterized all unmixed bipartite graphs independently. H. Haghighi in [2] has given a characterization for unmixed tripartite graphs under a certain condition, and recently R. Jafarpour-Golzari and R. Zaare-Nahandi in [5] have generalized Haghighi's result for unmixed  $r$ -partite graphs. On characterization of unmixed  $r$ -partite hypergraphs, almost no study has been done. Only in [6] a classification of a very special class of unmixed multipartite hypergraphs has been provided.

In this paper we give a characterization of all unmixed  $d$ -uniform  $r$ -partite hypergraphs under a certain condition which we name it (\*\*). Also we give necessary or sufficient conditions for unmixedness in more general classes of  $d$ -uniform hypergraphs (Propositions 3.6, 3.8).

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## 2. Preliminaries

In the sequel, we use [12] and [1] for terminology and notations on graphs and hypergraphs respectively.

Let  $G = (V, E)$  be a simple finite graph. For  $x, y \in V$ ,  $x \sim y$  means that  $x$  and  $y$  are adjacent. A subset  $M$  of  $V$  is said to be independent if for every  $x, y \in M$ ,  $x \not\sim y$ . A vertex cover for  $G$  is a subset  $C$  of  $V$  such that every edge of  $G$ , intersects  $C$ . A vertex cover  $C$  is minimal whenever there is no any pure subset of  $C$  which is a vertex cover.  $G$  is called unmixed if all minimal vertex cover of  $G$  have the same number of elements. A subset  $Q$  of  $V$  is said to be a clique if for every two distinct vertices  $x, y \in Q$ ,  $x \sim y$ .

A hypergraph  $\mathcal{H}$  on a finite nonempty set  $V$  is a set of nonempty subsets of  $V$  such that  $\bigcup_{e \in \mathcal{H}} e = V$ . The elements of  $V$  are called vertices and each element of  $\mathcal{H}$  is said a hyperedge. We denote by  $V(\mathcal{H})$  and  $E(\mathcal{H})$ , the sets of vertices and hyperedges of  $\mathcal{H}$  respectively. A hypergraph is said to be simple hypergraph or clutter if non of its two distinct hyperedges contains another. The hypergraph  $\mathcal{H}$  is called  $d$ -uniform (or  $d$ -graph), if all its hyperedges have the same cardinality  $d$ .

**Definition 2.1.** *An  $r$ -partite ( $r \geq 2$ ) hypergraph  $\mathcal{H}$ , is a hypergraph which  $V(\mathcal{H})$  can be partitioned to  $r$  subsets such that for every two vertices  $x, y$  in one part,  $x, y$  do not lie in any hyperedge. Such a partition of  $V(\mathcal{H})$  is called an  $r$ -partition of  $\mathcal{H}$ . If  $r = 2, 3$ , the  $r$ -partite hypergraph is said to be bipartite and tripartite respectively.*

In the hypergraph  $\mathcal{H}$ , two vertices  $x, y$  are said to be adjacent if there is a hyperedge containing  $x$  and  $y$ . We say that a hyperedge  $e$  is adjacent with a vertex  $x$  if  $x \in e$ . For a vertex  $x$  of  $\mathcal{H}$ , the neighborhood of  $x$ , denoted by  $N(x)$ , is the set of all vertices which are adjacent to  $x$ .

A subset  $M$  of  $V(\mathcal{H})$  is called independent if it dose not contain any hyperedge. An independent set  $M$  of  $\mathcal{H}$  is said to be maximal whenever it is not strictly contained in any other independent set. A subset  $C$  of  $V(\mathcal{H})$  is called a vertex cover, if every hyperedge of  $\mathcal{H}$  intersects it. A vertex cover is said to be minimal if there is no any pure subset of it which is also a vertex cover. It is clear that every maximal independent set is complement of a minimal vertex cover and vice versa.

**Definition 2.2.** *The hypergraph  $\mathcal{H}$  is said to be unmixed if all minimal vertex covers of  $\mathcal{H}$  have the same cardinality.*

A matching in a hypergraph  $\mathcal{H}$  is a set of hyperedges which are disjoint pairwise. A perfect matching is a matching such that every vertex of  $\mathcal{H}$  lies in at least one of its elements.

Let  $\{1, \dots, n\}$  is denoted by  $[n]$ . A simplicial complex on  $[n]$  is a set  $\Delta$  of subsets of  $[n]$  such that (a)  $\{x\} \in \Delta$ , for every  $x \in [n]$ , (b) if  $F \in \Delta$  and  $G \subseteq F$ , then  $G \in \Delta$ . Each element of  $\Delta$  is said to be a face. Dimension of the face  $F$ , denoted by  $\dim F$ , is defined as  $|F| - 1$  and dimension of  $\Delta$  is the maximum of dimensions of its faces.

Let  $\Delta$  be a simplicial complex on  $[n]$  and  $S$  be a nonempty set of subsets of  $[n]$  such that  $\bigcup_{N \in S} N = [n]$ . The simplicial complex generated by  $S$  is the set of all subsets of elements of  $S$ .

For a simplicial complex  $\Delta$  or a hypergraph  $\mathcal{H}$  on  $[n]$ , and for  $r \geq 0$ , the  $r$ -skeleton of  $\Delta$  or  $\mathcal{H}$ , is the set of all faces of  $\Delta$  whose dimension is at most  $r$  or all subsets of hyperedges of  $\mathcal{H}$  with cardinality not exceeding  $r + 1$ , respectively.

**Definition 2.3.** Let  $\mathcal{H}$  be a  $d$ -uniform ( $d \geq 2$ ) hypergraph. A  $(d - 1)$ -subset of a hyperedge is called a *submaximal edge*, and the set of all submaximal edges is denoted by  $SE(\mathcal{H})$ .

For  $\epsilon \in SE(\mathcal{H})$ , the set  $\{v \in V(\mathcal{H}) \mid \epsilon \cup \{v\} \in E(\mathcal{H})\}$ , is denoted by  $N(\epsilon)$ . If  $v \in N(\epsilon)$ , we write  $\epsilon \sim v$ .

In a  $d$ -uniform hypergraph  $\mathcal{H}$ , a clique is a subset  $W$  of  $V(\mathcal{H})$  such that every its subset of size  $d$ , is a hyperedge in  $\mathcal{H}$ .

### 3. Unmixed hypergraphs

Let  $\mathcal{H}$  is a  $d$ -uniform  $r$ -partite hypergraph with  $2 \leq d \leq r$ . We say that  $\mathcal{H}$  satisfies the condition  $(**)$  for  $r \geq 2$ , if  $\mathcal{H}$  can be partitioned to  $r$  parts  $V_i = \{x_{1i}, \dots, x_{ni}\}$ ,  $1 \leq i \leq r$ , such that  $\{x_{j1}, x_{j2}, \dots, x_{jr}\}$  is a clique for every  $1 \leq j \leq n$ .

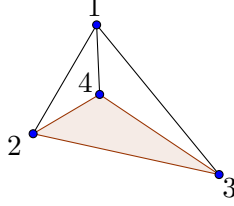
The authors in [5] have presented a necessary and sufficient condition for unmixedness of an  $r$ -partite graph which satisfies the following condition  $(*)$  for  $r \geq 2$ .

We say a graph  $G$  satisfies the condition  $(*)$  for an integer  $r \geq 2$ , if  $G$  can be partitioned to  $r$  parts  $V_i = \{x_{1i}, \dots, x_{ni}\}$ ,  $1 \leq i \leq r$ , such that for all  $1 \leq j \leq n$ ,  $\{x_{j1}, \dots, x_{jr}\}$  is a clique.

Let  $\mathcal{H}$  be a  $d$ -uniform  $r$ -partite hypergraph ( $2 \leq d \leq r$ ) on  $[n]$  which satisfies the condition  $(**)$  for  $r \geq 2$ . Then the 1-skeleton of  $\mathcal{H}$  is an  $r$ -partite graph which satisfies the condition  $(*)$  for  $r$ . But in general, the

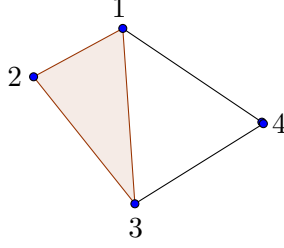
unmixedness of a hypergraph and its 1-skeleton, are two independent facts, as the following example exhibits.

**Example 3.1.** The following clutter is not unmixed while its 1-skeleton is unmixed as a graph.



Note that the sets  $\{2, 4, 3\}$  and  $\{1, 4\}$  are two minimal vertex covers with different sizes for the hypergraph.

Conversely, the following clutter is unmixed but its 1-skeleton is not.



Note that the sets  $\{1, 3\}$  and  $\{2, 3, 4\}$  are two minimal vertex covers with different sizes for the 1-skeleton.

This gives us a motivation for finding a necessary and sufficient condition under which a  $d$ -uniform  $r$ -partite ( $2 \leq d \leq r$ ) hypergraph satisfying the condition  $(**)$  for  $r$ , is unmixed.

First we prove a lemma.

**Lemma 3.2.** *Let  $\mathcal{H}$  be a  $d$ -uniform  $r$ -partite ( $2 \leq d \leq r$ ) hypergraph which satisfies the condition  $(**)$  for  $r$ . If  $\mathcal{H}$  is unmixed, then every minimal vertex cover of  $\mathcal{H}$  contains exactly  $r - d + 1$  elements of each clique  $\{x_{j1}, x_{j2}, \dots, x_{jr}\}$ .*

*Proof.* Let  $C$  be a minimal vertex cover of  $\mathcal{H}$ . For every  $1 \leq q \leq n$ ,  $C$  contains at least  $r - d + 1$  vertices of the clique  $\{x_{q1}, x_{q2}, \dots, x_{qr}\}$ ,

because if  $C$  contains at most  $r - d$  vertices of that clique, it does not cover hyperedges on remaining vertices. Therefore a vertex cover must contain at least  $n(r - d + 1)$  vertices. On the other hand, the set  $\bigcup_{i=1}^{r-d+1} V_i$  is a minimal vertex cover of  $\mathcal{H}$  with  $n(r - d + 1)$  vertices. This completes the proof.  $\square$

Now we present the main theorem of this paper.

**Theorem 3.3.** *Let  $\mathcal{H}$  be a  $d$ -uniform  $r$ -partite ( $2 \leq d \leq r$ ) hypergraph which satisfies the condition  $(**)$  for  $r$ . Then  $\mathcal{H}$  is unmixed if and only if the following condition holds.*

*For every  $1 \leq q \leq n$ , if  $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_{r-d+2}$  are submaximal edges such that*

$$\mathbf{e}_1 \sim x_{qi_1}, \mathbf{e}_2 \sim x_{qi_2}, \dots, \mathbf{e}_{r-d+2} \sim x_{qi_{r-d+2}}$$

*where  $i_1, i_2, \dots, i_{r-d+2}$  are distinct, then the set*

$$\mathbf{e}_1 \cup \mathbf{e}_2 \cup \dots \cup \mathbf{e}_{r-d+2}$$

*is not independent.*

*Proof.* Let  $\mathcal{H}$  be unmixed. We show that the mentioned condition holds. Suppose in contrary

$$\mathbf{e}_1 \sim x_{qi_1}, \mathbf{e}_2 \sim x_{qi_2}, \dots, \mathbf{e}_{r-d+2} \sim x_{qi_{r-d+2}}$$

where  $i_1, i_2, \dots, i_{r-d+2}$  are distinct but the set

$$F = \mathbf{e}_1 \cup \mathbf{e}_2 \cup \dots \cup \mathbf{e}_{r-d+2}$$

is independent. Therefore there is a maximal independent set  $M$  containing  $F$ . Since  $M$  is a maximal independent set,  $C := V(\mathcal{H}) \setminus M$  is a minimal vertex cover of  $\mathcal{H}$  which contains no any element of  $F$ . Since  $C$  is a vertex cover of  $\mathcal{H}$ ,  $C$  contains the vertices  $x_{qi_1}, x_{qi_2}, \dots, x_{qi_{r-d+2}}$ . But by Lemma 3.2,  $C$  contains exactly  $r - d + 1$  vertices of every clique, a contradiction.

Conversely, let the mentioned condition holds. We show that  $\mathcal{H}$  is unmixed. It is enough to show that every minimal vertex cover of  $\mathcal{H}$  contains exactly  $r - d + 1$  vertices of each clique  $\{x_{q1}, x_{q2}, \dots, x_{qr}\}$ . Let  $C$  be an arbitrary minimal vertex cover and  $1 \leq q \leq n$ . Then  $C$  intersects the set  $\{x_{q1}, x_{q2}, \dots, x_{qr}\}$  in at least  $r - d + 1$  elements. Let  $C$  intersects the mentioned clique in at least  $r - d + 2$  elements. Without loss of generality, we assume that this elements are  $x_{q1}, x_{q2}, \dots, x_{q(r-d+2)}$ . For each  $i$ ,  $1 \leq i \leq r - d + 2$ ,  $x_{qi}$  is in the minimal vertex cover  $C$ . Then there is a hyperedge  $e_i$  covered only by  $x_{qi}$ . That is,  $e_i \cap C = \{x_{qi}\}$ .

Suppose that  $\mathfrak{e}_i = e_i \setminus \{x_{qi}\}$ . Then the sets  $\mathfrak{e}_i$  are submaximal edges such that

$$\mathfrak{e}_1 \sim x_{qi_1}, \mathfrak{e}_2 \sim x_{qi_2}, \dots, \mathfrak{e}_{r-d+2} \sim x_{qi_{r-d+2}}$$

and  $\mathfrak{e}_1 \cup \mathfrak{e}_2 \cup \dots \cup \mathfrak{e}_{r-d+2}$  dose not intersects  $C$ . But by hypothesis

$$\mathfrak{e}_1 \cup \mathfrak{e}_2 \cup \dots \cup \mathfrak{e}_{r-d+2}$$

is not independent. That is, it contains a hypergraph  $e$  which is not covered by  $C$ , a contradiction.  $\square$

The following theorem of Villarreal on unmixedness of bipartite graphs can be concluded from the Theorem 3.3, where  $r = 2, d = 2$ .

**Corollary 3.4.** [11, Theorem 1.1] *Let  $G$  be a bipartite graph without isolated vertices. Then  $G$  is unmixed if and only if there is a bipartition  $V_1 = \{x_1, \dots, x_g\}, V_2 = \{y_1, \dots, y_g\}$  of  $G$  such that: (a)  $\{x_i, y_i\} \in E(G)$ , for all  $i$ , and (b) if  $\{x_i, y_j\}$  and  $\{x_j, y_k\}$  are in  $E(G)$ , and  $i, j, k$  are distinct, then  $\{x_i, y_k\} \in E(G)$ .*

The following theorem can be concluded from theorem 3.3, where  $d=2$ .

**Corollary 3.5.** [5, Theorem 2.3] *Let  $G$  be an  $r$ -partite graph which satisfies the condition  $(*)$  for  $r$ . Then  $G$  is unmixed if and only if the following condition hold:*

*For every  $1 \leq q \leq n$ , if there is a set  $\{x_{k_1 s_1}, \dots, x_{k_r s_r}\}$  such that*

$$x_{k_1 s_1} \sim x_{q1}, \dots, x_{k_r s_r} \sim x_{qr},$$

*then the set  $\{x_{k_1 s_1}, \dots, x_{k_r s_r}\}$  is not independent.*

Now we prove two propositions about  $d$ -uniform ( $d \geq 2$ ) hypergraphs by methods used in the proof of Theorem 3.3.

**Proposition 3.6.** *Let  $\mathcal{H}$  be a  $d$ -uniform ( $d \geq 2$ ) hypergraph on vertex set  $\{x_{ji} \mid 1 \leq j \leq n, 1 \leq i \leq d\}$  with perfect matching*

$$\{\{x_{j1}, x_{j2}, \dots, x_{jd}\} \mid 1 \leq j \leq n\}.$$

*If  $\mathcal{H}$  is unmixed and has a minimal vertex cover of size  $n$ , then for every  $1 \leq q \leq n$ , if  $\mathfrak{e}_1, \mathfrak{e}_2$  be two submaximal edges such that*

$$\mathfrak{e}_1 \sim x_{qi_1}, \mathfrak{e}_2 \sim x_{qi_2}$$

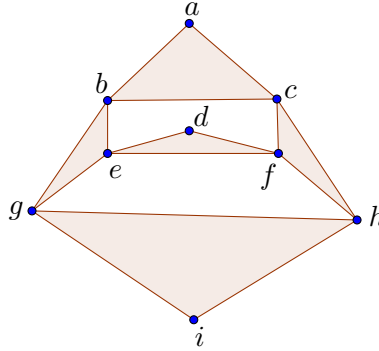
*where  $i_1, i_2$  are distinct, then the set  $\mathfrak{e}_1 \cup \mathfrak{e}_2$  is not independent.*

*Proof.* Let  $1 \leq q \leq n$  be arbitrary. Let in contrary the set  $\mathfrak{e}_1 \cup \mathfrak{e}_2$  is independent. Therefore  $\mathfrak{e}_1 \cup \mathfrak{e}_2$  is contained in a maximal independent set  $M$ . Set  $T = V(G) \setminus M$ .  $T$  is a minimal vertex cover and since it does not contain any element of  $\mathfrak{e}_1 \cup \mathfrak{e}_2$ , then  $T$  contains  $x_{qi_1}, x_{qi_2}$  and then  $T$  is at least of size  $n + 1$ , a contradiction.  $\square$

**Example 3.7.** In 3-uniform hypergraph

$$\mathcal{H} = \{\{a, b, c\}, \{d, e, f\}, \{g, h, i\}, \{b, g, e\}, \{c, f, h\}\},$$

we have the perfect matching  $\{\{a, b, c\}, \{d, e, f\}, \{g, h, i\}\}$  of size 3.



We show by proposition 3.7 that  $\mathcal{H}$  is not unmixed.

Let  $\mathcal{H}$  be unmixed (by contrary).  $\mathcal{H}$  has the minimal vertex cover  $\{b, e, h\}$  of size 3. Now we have 2 hyperedges  $\{a, b, c\}$  and  $\{b, g, e\}$  in relevance with the hyperedge  $\{a, b, c\}$  of perfect matching, but  $\{g, e, h, f\}$  is independent, a contradiction.

**Proposition 3.8.** Let  $\mathcal{H}$  is a  $d$ -uniform ( $d \geq 2$ ) hypergraph on the vertex set  $\{x_{ji} \mid 1 \leq j \leq n, 1 \leq i \leq d\}$  with perfect matching

$$\{\{x_{j1}, x_{j2}, \dots, x_{jd}\} \mid 1 \leq j \leq n\}.$$

Then a sufficient condition for unmixedness of  $\mathcal{H}$  is that for every  $1 \leq q \leq n$ , if  $\mathfrak{e}_1, \mathfrak{e}_2$  be two submaximal edges such that

$$\mathfrak{e}_1 \sim x_{qi_1}, \mathfrak{e}_2 \sim x_{qi_2}$$

where  $i_1, i_2$  are distinct, then  $\mathfrak{e}_1 \cup \mathfrak{e}_2$  is not independent.

*Proof.* Let  $\mathcal{H}$  satisfies the above condition. We show that  $\mathcal{H}$  is unmixed. It is enough to show that every minimal vertex cover of  $\mathcal{H}$  contains exactly one element of each hyperedge of the perfect matching. Let

$T$  be a minimal vertex cover.  $T$  contains at least one element of each hyperedge of perfect matching. Suppose in contrary that  $T$  chooses at least two elements from hyperedge  $\{x_{q1}, x_{q2}, \dots, x_{qd}\}$ . Without loss of generality, let the elements  $x_{q1}$  and  $x_{q2}$  are chosen. Since  $x_{q1}$  is in minimal vertex cover, then there exist at least  $d - 1$  distinct vertices in  $N(x_{q1})$ , such that they dose not belong to  $T$  and form a hyperedge together with  $x_{q1}$ . Name the set of these vertices  $\mathfrak{e}_1$ . We have

$$\mathfrak{e}_1 \sim x_{q1}.$$

Similarly, Since  $x_{q2}$  is in minimal vertex cover, with a similar argument, there is a submaximal edge  $\mathfrak{e}_2$  consisting of  $d - 1$  distinct vertices, no one belonging to  $T$ , such that

$$\mathfrak{e}_2 \sim x_{q2}.$$

Now  $\mathfrak{e}_1 \cup \mathfrak{e}_2$  dose not intersect  $T$ . But according to hypothesis  $\mathfrak{e}_1 \cup \mathfrak{e}_2$  is not independent. That is, it contains a hyperedge  $e$  which is not covered by  $T$ , a contradiction.  $\square$

#### 4. Edge ideal of unmixed hypergraphs

In this section, we provide an algebraic interpretation for Theorem 3.3.

**Definition 4.1.** Let  $\mathcal{H}$  be a hypergraph with  $V(\mathcal{H}) = \{x_1, \dots, x_m\}$ . Let  $K[x_1, \dots, x_m]$  be the polynomial ring with indeterminates  $x_1, \dots, x_m$  and coefficients in a field  $K$ . For a subset  $D = \{x_{i_1}, \dots, x_{i_r}\} \subseteq V(\mathcal{H})$ , let  $X_D = x_{i_1} \dots x_{i_r}$ . We define the edge ideal of  $\mathcal{H}$  to be

$$I(\mathcal{H}) := (X_e \mid e \in E(\mathcal{H})).$$

The quotient ring  $K[\mathcal{H}] := \frac{K[x_1, \dots, x_m]}{I(\mathcal{H})}$  is called the edge ring of  $\mathcal{H}$ .

Let  $R$  be a commutative ring. An element  $a \in R$  is called zero divisor if there is  $b \neq 0$  in  $R$  such that  $ab = 0$ .

**Theorem 4.2.** Let  $\mathcal{H}$  be a  $d$ -uniform  $r$ -partite ( $2 \leq d \leq r$ ) hypergraph which satisfies the condition  $(**)$  for  $r \geq 2$ . Then  $\mathcal{H}$  is unmixed if and only if for every  $1 \leq q \leq n$ , and every  $1 \leq i_1 < i_2 < \dots < i_{r-d+2} \leq r$ ,  $\bar{x}_{qi_1} + \bar{x}_{qi_2} + \dots + \bar{x}_{qi_{r-d+2}}$  is not a zero divisor in  $K[\mathcal{H}]$ . Here  $\bar{x}_{qi_t}$  denotes the image of  $x_{qi_t}$  in  $K[\mathcal{H}]$ .



*Proof.* Let  $\mathcal{H}$  be unmixed. If for some  $1 \leq q \leq n$  and some  $1 \leq i_1 < i_2 < \dots < i_{r-d+2} \leq r$ ,  $\bar{x}_{qi_1} + \bar{x}_{qi_2} + \dots + \bar{x}_{qi_{r-d+2}}$  is zero divisor in  $K[\mathcal{H}]$ , then there is a polynomial  $f \notin I(\mathcal{H})$  in

$$S = K[x_{11}, \dots, x_{n1}, x_{12}, \dots, x_{n2}, \dots, x_{1r}, \dots, x_{nr}]$$

such that  $f(x_{qi_1} + x_{qi_2} + \dots + x_{qi_{r-d+2}}) \in I(\mathcal{H})$ . The ideal  $I(\mathcal{H})$  is a monomial ideal and therefore we may assume that  $f$  is a monomial and then each monomial of must belong to  $I(\mathcal{H})$  (see [3]). That is, each monomial of the above polynomial must be divided by some generator of  $I(\mathcal{H})$  which comes from a hyperedge. Let  $x_{qi_t}$  be such a monomial. Then there is a hyperedge  $e_t$  in  $\mathcal{H}$  such that  $X_{e_t} | f x_{qi_t}$ . But  $X_{e_t} \nmid f$ . Then  $x_{qi_t} | X_{e_t}$  and  $e_t \setminus \{x_{qi_t}\}$  is a subminimal edge  $\mathfrak{e}_t$  and  $X_{\mathfrak{e}_t} | f$ . Therefore, for  $x_{qi_1}, x_{qi_2}, \dots, x_{qi_{r-d+2}}$ , There are  $r - d + 2$  submaximal edges  $\mathfrak{e}_1, \mathfrak{e}_2, \dots, \mathfrak{e}_{r-d+2}$  such that  $X_{\mathfrak{e}_t} | f$ , for  $1 \leq t \leq r - d + 2$ , and

$$\mathfrak{e}_1 \sim x_{qi_1}, \mathfrak{e}_2 \sim x_{qi_2}, \dots, \mathfrak{e}_{r-d+2} \sim x_{qi_{r-d+2}}.$$

Now by theorem 3.3, the set

$$\mathfrak{e}_1 \cup \mathfrak{e}_2 \cup \dots \cup \mathfrak{e}_{r-d+2}$$

contains a hyperedge  $e$ . Now  $X_{\mathfrak{e}_1 \cup \mathfrak{e}_2 \cup \dots \cup \mathfrak{e}_{r-d+2}} | f$ . Then  $X_e | f$ . Then  $f \in I(\mathcal{H})$ , a contradiction.

Conversely, let for every  $1 \leq q \leq n$ , and every  $1 \leq i_1 < i_2 < \dots < i_{r-d+2} \leq r$ ,  $\bar{x}_{qi_1} + \bar{x}_{qi_2} + \dots + \bar{x}_{qi_{r-d+2}}$  is not zero divisor in  $K[\mathcal{H}]$ . If  $\mathcal{H}$  is not unmixed, by theorem 3.3, there is an integer  $1 \leq q \leq n$ , and submaximal edges  $\mathfrak{e}_1, \mathfrak{e}_2, \dots, \mathfrak{e}_{r-d+2}$ , such that

$$\mathfrak{e}_1 \sim x_{qi_1}, \mathfrak{e}_2 \sim x_{qi_2}, \dots, \mathfrak{e}_{r-d+2} \sim x_{qi_{r-d+2}}.$$

where where  $i_1, i_2, \dots, i_{r-d+2}$  are distinct and

$$\mathfrak{e}_1 \cup \mathfrak{e}_2 \cup \dots \cup \mathfrak{e}_{r-d+2}$$

is an independent set. Set  $e_t = \mathfrak{e}_t \cup x_{qi_t}$ , for  $1 \leq t \leq r - d + 2$ .  $e_t$ 's are hyperedge. Let  $X = X_{\mathfrak{e}_1 \cup \mathfrak{e}_2 \cup \dots \cup \mathfrak{e}_{r-d+2}}$ .  $X$  is not in  $I(\mathcal{H})$  but  $X.(x_{qi_1} + x_{qi_2} + \dots + x_{qi_{r-d+2}}) \in I(\mathcal{H})$ , a contradiction.  $\square$

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